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ON HYPERSINGULAR INTEGRALS AND CERTAIN SPACES OF LOCALLY DIFFERENTIABLE FUNCTIONS

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1. **Introduction.** In this paper, we shall study relations between *pointwise* convergence of hypersingular integrals and *local* differential properties of functions. Our results will partly generalize a theorem of Calderón and Zygmund and an unpublished theorem of E. M. Stein.

We will use standard notation for points and functions in n-dimensional Euclidean space E^n, $n \geqq 2$. If $f(x) \in L^p(E^n)$, $1 \leqq p < \infty$, set

$$\tilde{f}_\varepsilon(x) = \int_{|z|>\varepsilon} [f(x-z)-f(x)] \frac{\Omega(z')}{|z|^{n+\alpha}}\, dz$$

for $\varepsilon > 0$ and $0<\alpha<2$, where Ω is a bounded real-valued function homogeneous of degree zero which satisfies

$$\int_\Sigma z_j'\Omega(z')\, dz' = 0 \qquad (j = 1,\ldots,n) \tag{1.1}$$

for $1 \leqq \alpha < 2$. Here Σ denotes the unit sphere of points $z'=z/|z|$, $z\neq 0$.

If $\lim_{\varepsilon\to 0}\tilde{f}_\varepsilon(x)$ exists in some sense, we call it a hypersingular integral of f. If, for example, f satisfies the global differentiability condition $f \in L^p_\alpha = J^\alpha L^p$ (see [2]), the convergence of $\tilde{f}_\varepsilon$ in various senses was studied in [8], [14] and [15]. Thus far, however, the convergence of $\tilde{f}_\varepsilon$ for f satisfying a local differentiability condition has been studied for $n \geqq 2$ only in case $\alpha = 1$. (See [5].)

Following [4], we say an $f \in L^p$, $1 \leqq p < \infty$, belongs to $t^p_\alpha(x_0)$ if there is a polynomial $P_{x_0}(z)$ of degree less than or equal to α such that

$$\left(\varepsilon^{-n}\int_{|z|<\varepsilon} |f(x_0+z)-P_{x_0}(z)|^p\, dz\right)^{1/p} = o(\varepsilon^\alpha)$$

as $\varepsilon \to 0$. We say $f \in T^p_\alpha(x_0)$ if there is a polynomial of degree strictly less than α such that

$$\left(\varepsilon^{-n}\int_{|z|<\varepsilon} |f(x_0+z)-P_{x_0}(z)|^p\, dz\right)^{1/p} = O(\varepsilon^\alpha)$$

for $\varepsilon > 0$.

Given $0<\alpha<2$ let $\beta = [\alpha]+1-\alpha$, so that $\alpha+\beta = 1$ if $0<\alpha<1$ and $\alpha+\beta=2$ if $1 \leqq \alpha < 2$. Roughly speaking, our main result is that for $f \in T^p_\alpha(x)$ the convergence

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of $\tilde{f}_\varepsilon(x)$ is equivalent almost everywhere to the condition $J^\beta f \in T^p_{\alpha+\beta}(x)$. Calderón and Zygmund show in [4] (Theorems 4 and 5) that if $f \in T^p_\alpha(x)$ for $x \in E$ then $J^\gamma f \in T^p_{\alpha+\gamma}(x)$ for almost every $x \in E$, except in the special case that $\alpha+\gamma$ is an integer but α and γ are not. This is precisely our case, however, and an example of the complications which may arise can be found in [16, pp. 136–138].

We shall prove the following results:

THEOREM 1. *Given* $0<\alpha<2$, *let* Ω *be a bounded function homogeneous of degree zero which satisfies* (1.1) *when* $1 \leqq \alpha < 2$. *Let* $f \in L^p$, $1 \leqq p < \infty$, $E \subset E^n$ *and* $\beta = [\alpha] + 1 - \alpha$. *If* $f \in T^p_\alpha(x)$ *for* $x \in E$ *and* $J^\beta f \in T^p_{\alpha+\beta}(x)$ *for* $x \in E$ *then*

$$\lim_{\varepsilon \to 0} \int_{|z|>\varepsilon} [f(x-z)-f(x)] \frac{\Omega(z')}{|z|^{n+\alpha}}\, dz$$

exists and is finite for almost all $x \in E$.

Conversely,

THEOREM 2. *Let* $f \in L^p$, $1 \leqq p < \infty$, $E \subset E^n$, $0<\alpha<2$. *Suppose* $f \in T^p_\alpha(x)$ *for* $x \in E$ *and each*

$$\int_{|z|>\varepsilon} [f(x-z)-f(x)] \frac{\Omega_j(z)}{|z|^{n+\alpha}}\, dz$$

converges for $x \in E$, *where* $\{\Omega_j\}$ *is a basis for the spherical harmonics of a fixed degree* $m \geqq 0$, $m \neq 1$ *when* $1 \leqq \alpha < 2$ *and* $m=0$ *when* $p=1$. *Then with* $\beta = [\alpha]+1-\alpha$, $J^\beta f \in T^p_{\alpha+\beta}(x)$ *for almost all* $x \in E$.

When $\alpha=1$, the hypothesis $f \in T^p_1(x)$ for $x \in E$ implies that $f \in t^p_1(x)$ and $J^1 f \in t^p_2(x)$ for almost all $x \in E$(1) and Theorem 1 is a known result of Calderón and Zygmund [5]. Also, Theorem 2 for $\alpha=1$ is vacuous and a replacement result is the following.

THEOREM 3. *Let* $f \in L^p$, $1 \leqq p < \infty$, $E \subset E^n$. *If*

$$\left(\varepsilon^{-n} \int_{|z|<\varepsilon} |f(x+z)+f(x-z)-2f(x)|^p\, dz\right)^{1/p} = O(\varepsilon)$$

for $x \in E$ *and each*

$$\int_{|z|>\varepsilon} [f(x-z)-f(x)] \frac{\Omega_j(z)}{|z|^{n+1}}\, dz$$

converges for $x \in E$, *where the* Ω_j *are as in Theorem* 2, *then* $f \in t^p_1(x)$ *for almost all* $x \in E$.

Theorem 3 for $\Omega \equiv 1$ was proved independently by E. M. Stein. It turns out that for all $0<\alpha<2$ one can replace the condition $f \in T^p_\alpha(x)$ of Theorem 2 by an apparently weaker condition. See the remark at the end of §4.

(1) See Theorem 5 of [4]. Although Theorems 4 and 5 are stated for $p>1$, it is not hard to see they remain true for $p=1$ when, with the notation of [4], $q=1$ and $u>0$.

Although we stated our theorems for $n \geqq 2$ they have analogues for $n=1$ which are related to the results of [13]. In their present form, our results do not include those of Sagher [7] for hypersingular integrals with complex homogeneity.

We shall prove Theorem 1 in §2, Theorem 2 in §3 and Theorem 3 in §4. §4 also contains an apparent improvement of Theorem 2.

2. **Proof of Theorem 1.** We will use the method in [11] to prove Theorem 1. We need a long list of lemmas, and in order to shorten their presentation we will assume $1<\alpha<2$ whenever convenient. We also note that it suffices to prove Theorem 1 for $p=1$ since we may assume E is bounded and f has compact support and since the condition $f \in T_\alpha^p(x)$ for $p>1$ implies $f \in T_\alpha^1(x)$.

We recall that $f \in L_\alpha^p$, $1 \leqq p<\infty$, $\alpha>0$, if $f=J^\alpha\phi=G_\alpha * \phi$ for $\phi \in L^p$ where G_α is a positive integrable function with the following properties (see e.g. [4]):

(a) $\hat{G}_\alpha(x)=(1+|x|^2)^{-\alpha/2}$,

(b) G_α is infinitely differentiable except at $x=0$ and for $x \neq 0$, $0<\alpha<n$ and $|\nu| \geqq 0$

$$|(\partial^\nu/\partial x^\nu)G_\alpha(x)| \leqq c_{\alpha,\nu}e^{-|x|}[1+|x|^{-n-|\nu|+\alpha}].$$

Lemma 1. *Let $f \in L_\alpha^p$ for some $0<\alpha<2$ and let Ω satisfy the hypothesis of Theorem 1. Then for $1 \leqq p<\infty$*

$$\tilde{f}(x) = \lim_{\varepsilon\to 0}\int_{|z|>\varepsilon}[f(x-z)-f(x)]\frac{\Omega(z')}{|z|^{n+\alpha}}\,dz$$

exists and is finite for almost all x, and for $1<p<\infty$, the transformation $f \to \tilde{f}$ sends L_α^p boundedly into L^p.

For a proof, see [15].

Lemma 2. *Given $\lambda>0$,*

(a) $|x|^\lambda=(1+|x|^2)^{\lambda/2}\,d\hat{\mu}(x)$,

(b) $(1+|x|^2)^{\lambda/2}=|x|^\lambda\,d\hat{\sigma}(x)+d\hat{\tau}(x)$,

where $d\hat{\mu}$ is the sum of 1, a finite linear combination of terms $\hat{G}_{2k}$, $k=1,2,\dots$, and the Fourier transform of a function with derivatives up to a preassigned order belonging to all L^p, $1 \leqq p \leqq \infty$, $d\hat{\sigma}$ is the sum of 1 and a finite linear combination of terms $\hat{G}_{2k}$, $k=1,2,\dots$, and $d\hat{\tau}$ is the Fourier transform of a function with derivatives up to a preassigned order belonging to all L^p, $1 \leqq p \leqq \infty$.

Parts (a) and (b) of Lemma 2 are stated in [8]. The proof of the rest of the lemma is not difficult and we omit it.

Lemma 3. *Let $f \in L^1$, $1<\alpha<2$, $\alpha+\beta=2$, $F=J^\beta f$. Then for almost all x,*

$$f(x) = cF_\tau(x)+c_\beta\int[F_\sigma(x+z)-F(x)]\frac{dz}{|z|^{n+\beta}} \tag{2.1}$$

where $F_\sigma=F * d\sigma$, $F_\tau=F * d\tau$, $d\sigma$ and $d\tau$ being defined by Lemma 2 with $\lambda=\beta$.

The integral in (2.1) exists almost everywhere in the principal value sense by Lemma 1 since $F_\sigma \in L^1_\beta$. Moreover, by Lemma (1.6) of [14],

$$c_\beta \int_{|z|>\varepsilon} [F_\sigma(x+z)-F_\sigma(x)] \frac{dz}{|z|^{n+\beta}} - \int_{E^n} F_\sigma(x+z)[|z|^\beta e^{-\varepsilon|z|}]^\wedge \, dz$$

tends to zero with ε for almost all x. Hence the right side of (2.1) is the limit almost everywhere of

$$\begin{aligned}\int_{E^n} F_\tau(x+z)[e^{-\varepsilon|z|}]^\wedge \, dz + \int_{E^n} F_\sigma(x+z)[|z|^\beta e^{-\varepsilon|z|}]^\wedge \, dz \\ = c \int \hat{F}(z)[|z|^\beta \, d\hat{\sigma}(z) + d\hat{\tau}(z)] e^{i(x\cdot z)} e^{-\varepsilon|z|} \, dz \\ = c \int \hat{f}(z) e^{i(x\cdot z)} e^{-\varepsilon|z|} \, dz\end{aligned}$$

by Lemma 2(b) and the fact that $\hat{F}(z) = (1+|z|^2)^{-\beta/2}\hat{f}(z)$. The last integral is essentially the Poisson integral of f and converges to a constant times f almost everywhere.

LEMMA 4. *If $f \in L^1$ and $\alpha > 0$ is not an integer then $J^\alpha f \in t^1_\alpha(x)$ for almost all x.*

The proof of Lemma 4 is almost identical to that of Theorem 4 of [4]. Although the case $p=1$ is not considered there, the proof easily yields Lemma 4. (See also the proof of Lemma 4 of §3 below.)

LEMMA 5. *Let $1<\alpha<2$ and $v(x)$ and its first order derivatives be continuous and have compact support. For any $j=1,\ldots,n$,*

$$u(x) = \int_{E^n} v(x-z) \frac{z_j'}{|z|^{n-(\alpha-1)}} \, dz$$

belongs to $T^1_\alpha(x)$ uniformly in x.

Proof. If $u_i = (\partial/\partial x_i)u$ then

$$u_i(x) = \int_{E^n} v_i(x-z) \frac{z_j'}{|z|^{n-(\alpha-1)}} \, dz$$

is continuous and

$$\begin{aligned}|u_i(x+y)-u_i(x)| \leqq c \int \left| \frac{1}{|z+y|^{n-(\alpha-1)}} - \frac{1}{|z|^{n-(\alpha-1)}} \right| dz \\ + c \int \frac{1}{|z|^{n-(\alpha-1)}} |(z+y)_j' - z_j'| \, dz.\end{aligned}$$

Each integral is easily seen to be $O(|y|^{\alpha-1})$ and the lemma follows from Taylor's formula.

The remaining lemmas are taken from [4].

LEMMA 6. *Let P be a closed subset of E^n and U be the neighborhood of P of all points whose distance from P is less than* 1. *Then there is a covering of $U-P$ by nonoverlapping closed cubes K_m with $c^{-1} \leqq d_m/e_m \leqq c$, $0<c<\infty$, where e_m is the edge length of K_m and d_m is the distance from K_m to P.*

See Lemma (3.1) of [4].

LEMMA 7. *Let P be a compact set and $\delta(x)$ be the distance from x to P, with $\delta(x)=0$ for large x. Given $\lambda>0$*

$$\int_{E^n} \frac{\delta^\lambda(x+z)}{|z|^{n+\lambda}}\, dz \tag{2.2}$$

is finite for almost all $x \in P$.

LEMMA 8. *Let $F \in t_2^1(x)$ for $x \in E$, E a bounded measurable set. Given $\varepsilon>0$ there is a closed set $P \subset E$, $|E-P|<\varepsilon$, and a decomposition $F=G+H$ where G has two continuous derivatives and compact support, $H(x)=0$ for $x \in P$ and*

$$\int_{|z|<\varepsilon} |H(x+z)|\, dz \leqq M\varepsilon^{n+2}$$

uniformly for $x \in P$. Moreover, given $0<\lambda\leqq 2$,

$$\int \frac{|H(z)|}{|x-z|^{n+\lambda}\delta(z)^{2-\lambda}}\, dz \tag{2.3}$$

is finite for almost all $x \in P$, $\delta(z)$ being the distance from z to P.

Integration in (2.3) is of course extended over the complement of P. Lemma 8 for $\lambda=2$ is proved in [4, p. 189–190], and the proof for $0<\lambda\leqq 2$ is similar.

LEMMA 9. *Let $h \in T_\alpha^1(x)$, $1<\alpha<2$, uniformly for x in a closed set P, i.e.,*

$$\varepsilon^{-n} \int_{|z|<\varepsilon} \left| h(x+z)-h(x)-\sum z_j h_j(x) \right| dz \leqq M\varepsilon^\alpha$$

for $x \in P$. Then for x and $x+z$ in P,

$$\left| h(x+z)-h(x)-\sum z_j h_j(x) \right| \leqq M'|z|^\alpha$$

and

$$|h_j(x+z)-h_j(x)| \leqq M'|z|^{\alpha-1} \qquad (j=1,\ldots,n).$$

We can now prove Theorem 1 for $1<\alpha<2$. Let f and $F=J^\beta f$, $\beta=2-\alpha$, satisfy the hypothesis of Theorem 1 for $p=1$. Let F_σ and F_τ be defined as in Lemma 3. Since $F \in L^1$ and F_τ is a convolution of F with a function with bounded derivatives up to a preassigned order (Lemma 2), we may assume F_τ has bounded continuous second order derivatives everywhere. In particular,

$$\left|F_\tau(x-z)-F_\tau(x)+\sum z_j\left(\frac{\partial F_\tau}{\partial x_j}\right)(x)\right| \leqq M|z|^2, \tag{2.4}$$

for all x and z, $M<\infty$. Since $1<\alpha<2$,

$$\int \left[F_\tau(x-z)-F_\tau(x)+\sum z_j\left(\frac{\partial F_\tau}{\partial x_j}\right)(x)\right] \frac{\Omega(z')}{|z|^{n+\alpha}}\, dz$$

converges absolutely everywhere. Since Ω is orthogonal to polynomials of degree 1,

$$\int_{|z|>\varepsilon} [F_\tau(x-z)-F_\tau(x)] \frac{\Omega(z')}{|z|^{n+\alpha}}\, dz$$

converges everywhere as $\varepsilon \to 0$.

Hence, applying (2.1), it remains to prove the conclusion of Theorem 1 with f replaced by

$$\text{p.v.} \int [F_\sigma(x+z)-F_\sigma(x)] \frac{dz}{|z|^{n+\beta}}. \tag{2.5}$$

By (2.4) $F_\tau \in T_\alpha^1(x)$ everywhere and by (2.1) again, the same is true for $x \in E$ of (2.5). By Lemma 2, F_σ is the sum of F and a finite linear combination of terms $J^{2k}F$, $k \geqq 1$. It follows that $F_\sigma \in t_2^1(x)$ for almost all $x \in E$. Here we use first the fact, noted in §1, that $T_2^1(x)$ and $t_2^1(x)$ are equivalent almost everywhere and next the fact that $J^{2k}F=J^{2k+\beta}f \in t_{2k+\beta}^1(x)$ for almost all x (Lemma 4). Since $k \geqq 1$,

$$t_{2k+\beta}^1(x) \subset t_2^1(x).$$

Collecting these facts, we see it is enough to prove Theorem 1 for $f \in T_\alpha^1(x)$, $x \in E$, of the form

$$f(x) = \text{p.v.} \int_{E^n} [F(x+z)-F(x)] \frac{dz}{|z|^{n+\beta}},$$

where $F \in L_\beta^1$ and $F \in t_2^1(x)$ for $x \in E$. For such F, form the decomposition $F=G+H$ of F relative to a closed set $P \subset E$ (Lemma 8). We may assume $f \in T_\alpha^1(x)$ uniformly for $x \in P$. Consider

$$\begin{aligned}\int_{|z|>\varepsilon} [G(x+z)-G(x)] \frac{dz}{|z|^{n+\beta}} &= \frac{1}{\alpha-2}\int_\varepsilon^\infty \frac{d}{dt}(t^{\alpha-2})\, dt \int_\Sigma [G(x+tz')-G(x)]\, dz' \\ &= \frac{1}{\alpha-2}\left(t^{\alpha-2}\int_\Sigma [G(x+tz')-G(x)]\, dz'\,\Big|_\varepsilon^\infty \right. \\ &\qquad \left. -\sum_{j=1}^n \int_\varepsilon^\infty \frac{dt}{t^{2-\alpha}} \int_\Sigma z_j' G_j(x+tz')\, dz'\right),\end{aligned}$$

where $G_j=(\partial/\partial x_j)G$. At $t=\infty$ the integrated term is zero since $\alpha-2<0$. At $t=\varepsilon$ it is $O(\varepsilon^{\alpha-1})=o(1)$. Hence

$$g(x) = \lim_{\varepsilon\to 0}\int_{|z|>\varepsilon}[G(x+z)-G(x)]\frac{dz}{|z|^{n+\beta}}$$
(2.6)
$$= \frac{1}{\alpha-2}\sum_{j=1}^{n}\int_{E^n} G_j(x-z)\frac{z'_j}{|z|^{n-(\alpha-1)}}.$$

By Lemma 5, $g\in T^1_\alpha(x)$ uniformly in x for all x. Hence $h=f-g\in T^1_\alpha(x)$ uniformly for $x\in P$. Moreover,

$$h(x) = f(x)-g(x) = \lim_{\varepsilon\to 0}\int_{|z|>\varepsilon}[H(x+z)-H(x)]\frac{dz}{|z|^{n+\beta}}$$

almost everywhere. Since $H=0$ in P and (2.3) with $\lambda=2$ is finite for almost all $x\in P$,

$$h(x) = \int_{E^n}\frac{H(x+z)}{|z|^{n+\beta}}\,dz \tag{2.7}$$

for almost all $x\in P$, the integral converging absolutely.

To prove Theorem 1, it suffices to show that both

$$\tilde{g}_\varepsilon(x) = \int_{|z|>\varepsilon}[g(x-z)-g(x)]\frac{\Omega(z')}{|z|^{n+\alpha}}\,dz$$

and

$$\tilde{h}_\varepsilon(x) = \int_{|z|>\varepsilon}[h(x-z)-h(x)]\frac{\Omega(z')}{|z|^{n+\alpha}}\,dz$$

converge for almost all $x\in P$. Consider first $\tilde{g}_\varepsilon$. Since $G\in L^2_2=J^\alpha L^2_\beta$, it follows from (2.6) and Lemma 1 with $\Omega\equiv 1$ that $g\in L^2_\alpha$. By Lemma 1 again, $\tilde{g}_\varepsilon(x)$ converges for almost all x.

Turning to $\tilde{h}_\varepsilon$, we have $h\in T^1_\alpha(x)$, uniformly for $x\in P$, i.e., for $x\in P$

$$\int_{|z|<\varepsilon}\left|h(x+z)-h(x)-\sum z_jh_j(x)\right|dz \leqq M\varepsilon^{n+\alpha}$$

for certain $h_j(x)$. We claim that

$$h_j(x) = \int_{E^n} H(x-z)\frac{\partial}{\partial z_j}\left(\frac{1}{|z|^{n+\beta}}\right)dz \tag{2.8}$$

for almost all $x\in P$. Observe that the integral in (2.8) converges absolutely for almost all $x\in P$ since $\alpha>1$ (see (2.3) with $\lambda=2$). By Lemma 9, $h_j(x)$ is the derivative of h with respect to x_j restricted to P, i.e., if $\varepsilon_j=(0,\ldots,0,\varepsilon,0,\ldots,0)$ with ε as the jth entry, then

$$(h(x+\varepsilon_j)-h(x))/\varepsilon \to h_j(x)$$

as $\varepsilon \to 0$ provided x, $x+\varepsilon_j \in P$. On the other hand since we may assume (2.7) holds for all $x \in P$, we have for x and $x+\varepsilon_j$ in P

$$\frac{h(x+\varepsilon_j)-h(x)}{\varepsilon} - \int_{E^n} H(x-z)\frac{\partial}{\partial z_j}\left(\frac{1}{|z|^{n+\beta}}\right) dz$$
$$= \frac{1}{\varepsilon}\int_{E^n} H(x-z)\left[\frac{1}{|z+\varepsilon_j|^{n+\beta}} - \frac{1}{|z|^{n+\beta}} - \varepsilon\frac{\partial}{\partial z_j}\frac{1}{|z|^{n+\beta}}\right] dz$$
$$= \frac{1}{\varepsilon}\int_{|z|<2\varepsilon} + \frac{1}{\varepsilon}\int_{|z|>2\varepsilon} = A_\varepsilon + B_\varepsilon.$$

By the mean-value theorem,

$$|B_\varepsilon| \leqq c\varepsilon \int_{|z|>2\varepsilon} |H(x+z)| \frac{dz}{|z|^{n+\beta+2}}.$$

If $R(t) = \int_{|z|<t} |H(z+z)|\, dz$, then $R(t) \leqq Mt^{n+2}$ by Lemma 8 and

$$|B_\varepsilon| \leqq c\varepsilon \int_{2\varepsilon}^{\infty} \frac{dR(t)}{t^{n+\beta+2}}.$$

Integrating by parts, $B_\varepsilon = O(\varepsilon^{\alpha-1}) = o(1)$.

Even simpler estimates show that the terms

$$\frac{1}{\varepsilon}\int_{|z|<2\varepsilon} H(x+z)\frac{dz}{|z|^{n+\beta}} \quad \text{and} \quad \int_{|z|<2\varepsilon} H(x+z)\frac{\partial}{\partial z_j}\left(\frac{1}{|z|^{n+\beta}}\right) dz$$

of A_ε tend to zero with ε. The remaining term of A_ε is majorized by

$$\frac{1}{\varepsilon}\int_{|z|<3\varepsilon} |H(x+\varepsilon_j+z)| \frac{dz}{|z|^{n+\beta}} = \frac{1}{\varepsilon}\int_0^{3\varepsilon} \frac{dR_\varepsilon(t)}{t^{n+\beta}},$$

where $R_\varepsilon(t) = \int_{|z|<t} |H(x+\varepsilon_j+z)|\, dz \leqq Mt^{n+2}$ uniformly in ε. That (2.8) holds for almost all $x \in P$ now follows by integrating by parts.

We claim next that

$$\int_{E^n} \left|h(x+z)-h(x)-\sum z_j h_j(x)\right| \frac{dz}{|z|^{n+\alpha}} < \infty \tag{2.9}$$

for almost all $x \in P$. Since Ω is bounded and orthogonal to polynomials of degree 1, $\lim_{\varepsilon\to 0} \tilde{h}_\varepsilon(x)$ exists wherever (2.9) holds. If we assume that (2.7) and (2.8) hold for all $x \in P$, it is enough to show that (2.9) holds for each point of density x of P at which (2.2) is finite for $\lambda = \alpha$ and $\lambda = 1$ and at which (2.3) is finite for $\lambda = \alpha$ and $\lambda = 2$. Let $x = 0$ be such a point. Then (2.9) for $x = 0$ will follow if

$$\int_{|z|<\eta} \left|h(z)-h(0)-\sum z_j h_j(0)\right| \frac{dz}{|z|^{n+\alpha}} \tag{2.10}$$

is finite for some $\eta > 0$. In what follows we will denote by c a constant, possibly different in different occurrences, depending only on α and n.

Consider first that part of (2.10) with integration extended only over P. Applying (2.7), (2.8) and interchanging the order of integration,

$$\int_P \left|h(z)-h(0)-\sum z_j h_j(0)\right| \frac{dz}{|z|^{n+\alpha}}$$
$$\leqq \int |H(y)|\,dy \int_P \left|\frac{1}{|y-z|^{n+\beta}}-\frac{1}{|y|^{n+\beta}}+\sum z_j \frac{\partial}{\partial y_j}\left(\frac{1}{|y|^{n+\beta}}\right)\right| \frac{dz}{|z|^{n+\alpha}}.$$

By the mean-value theorem, the inner integral extended over $|z|<|y|/2$ is majorized by a constant times

$$\int_{|z|<|y|/2} \frac{|z|^2}{|y|^{n+\beta+2}} \frac{dz}{|z|^{n+\alpha}} = O(|y|^{-n-2}).$$

Since (2.3) with $\lambda=2$ and $x=0$ is finite, we may consider the inner integral above extended over $|z|>|y|/2$. Since $\alpha>1$,

$$\int |H(y)|\,dy \int_{|z|>|y|/2} \left|z_j \frac{\partial}{\partial y_j}\left(\frac{1}{|y|^{n+\beta}}\right)\right| \frac{dz}{|z|^{n+\alpha}}$$
$$\leqq c\int \frac{|H(y)|}{|y|^{n+\beta+1}}\,dy \int_{|z|>|y|/2} \frac{dz}{|z|^{n+\alpha-1}} = c\int \frac{|H(y)|}{|y|^{n+2}}\,dy.$$

The part

$$\int |H(y)|\,dy \int_{|z|>|y|/2} \frac{1}{|y|^{n+\beta}} \frac{dz}{|z|^{n+\alpha}}$$

can be treated similarly. Since $H=0$ in P and $|z-y|\geqq \delta(y)$ for $z\in P$ and $y\in P'$, the remaining part

$$\int |H(y)|\,dy \int_{P;|z|>|y|/2} \frac{1}{|y-z|^{n+\beta}} \frac{dz}{|z|^{n+\alpha}} \leqq c\int_{P'} \frac{|H(y)|}{|y|^{n+\alpha}}\,dy \int_{|y-z|>\delta(y)} \frac{dz}{|y-z|^{n+\beta}}$$
$$\leqq c\int \frac{H(y)}{|y|^{n+\alpha}\,\delta(y)^{2-\alpha}}\,dy < \infty.$$

Now consider the part of (2.10) with integration extended over P'. For any z, write $\omega(z)=h(z)-h(0)-\sum z_j h_j(0)$. With the notation of Lemma 6, let $p_m\in P$ be a point whose distance from each point of K_m is less than a constant (independent of m) times d_m. It is enough to show both

$$\sum_m \int_{K_m} |\omega(z)-\omega(p_m)|\,\frac{dz}{|z|^{n+\alpha}} \tag{2.11}$$

and

$$\sum_m \int_{K_m} |\omega(p_m)|\,\frac{dz}{|z|^{n+\alpha}} \tag{2.12}$$

are finite, summations being extended over all m for which K_m intersects $\{z : |z|<\eta\}$.

Let δ_m be the distance from K_m to 0. Since 0 is a point of density of P we can choose η so small that $|p_m|\leqq c\delta_m$ and $\delta_m\leqq |z|\leqq c\delta_m$ for $z\in K_m$, with c independent of m. Fix m and write $K=K_m$, $p=p_m$, etc. A term of (2.11) is then majorized by a constant independent of m times

$$\delta^{-n-\alpha}\int_K \left| h(z)-h(p)-\sum_j (z_j-p_j)h_j(p)\right| dz + d\delta^{-n-\alpha}\sum_j \int_K |h_j(p)-h_j(0)|\, dz.$$

If we replace integration over K by integration over $|z-p|<cd$, we only increase this. Moreover, $h\in T^1_\alpha(p)$ uniformly for $p\in P$ and $|h_j(p)-h_j(0)|\leqq c\delta^{\alpha-1}$ by Lemma 9. Hence the expression above is bounded by a constant independent of m times

$$d^{n+\alpha}/\delta^{n+\alpha}+d^{n+1}/\delta^{n+1}.$$

Since $|K|\geqq cd^n$ and $\delta(z)\geqq d$ for $z\in K$,

$$\frac{d^{n+\alpha}}{\delta^{n+\alpha}} \leqq c\int_K \frac{\delta^\alpha(z)}{\delta^{n+\alpha}}\,dz \leqq c\int_K \frac{\delta^\alpha(z)}{|z|^{n+\alpha}}\,dz.$$

Treating d^{n+1}/δ^{n+1} in the same way and summing over m, we see (2.11) is finite.

Turning to (2.12) we have

$$\tag{2.13} \begin{aligned}\int_K |\omega(p)|\,\frac{dz}{|z|^{n+\alpha}} \\ \leqq \int_K \frac{dz}{|z|^{n+\alpha}}\int |H(y)|\left|\frac{1}{|y-p|^{n+\beta}}-\frac{1}{|y|^{n+\beta}}-\sum p_j\frac{\partial}{\partial y_j}\left(\frac{1}{|y|^{n+\beta}}\right)\right| dy.\end{aligned}$$

The part of (2.13) with integration in the inner integral restricted to $|y|>2|p|$ is majorized by a constant times

$$\int_K \frac{dz}{|z|^{n+\alpha}}\int_{|y|>2|p|} |H(y)|\,\frac{|p|^2}{|y|^{n+\beta+2}}\,dy.$$

For $z\in K$, $|z|$ and $|p|$ are comparable since both are comparable to δ. Hence the last integral is less than a constant times

$$\int_K \frac{dz}{|z|^{n+\alpha}}\int_{|y|>c|z|} |H(y)|\,\frac{|z|^2}{|y|^{n+\beta+2}}\,dy.$$

Summing over m and changing the order of integration, we obtain

$$\int \frac{|H(y)|}{|y|^{n+\beta+2}}\,dy\int_{|z|<|y|/c}\frac{dz}{|z|^{n-\beta}} \leqq c\int \frac{|H(y)|}{|y|^{n+2}}\,dy.$$

Consider then the part of (2.13) with integration in the inner integral extended over $|y|<2|p|$. The parts

$$\int_K \frac{dz}{|z|^{n+\alpha}}\int_{|y|<2|p|}\frac{|H(y)|}{|y|^{n+\beta}}\,dy$$

and

$$\int_K \frac{dz}{|z|^{n+\alpha}}\int_{|y|<2|p|} |H(y)|\,\frac{|p|}{|y|^{n+\beta+1}}\,dy$$

can be handled as above—that is, by replacing $|p|$ by $|z|$, summing over m and interchanging the order of integration.

Consider finally the part

$$\int_K \frac{dz}{|z|^{n+\alpha}} \int_{|y|<2|p|} |H(y)| \frac{dy}{|p-y|^{n+\beta}}. \tag{2.14}$$

Let $\bar{K}=\bar{K}_m$ be K expanded concentrically k times, k taken large and independent of m. The part of (2.14) with inner integration over $\bar{K}$ is less than

$$\int_K \frac{dz}{|z|^{n+\alpha}} \int_{|p-y|<cd} \frac{|H(y)|}{|p-y|^{n+\beta}} dy = O\left(\frac{d^n}{\delta^{n+\alpha}}\right) \int_0^{cd} \frac{dR(t)}{t^{n+\beta}}$$

where $R(t)=\int_{|z|<t} |H(p+z)|\, dz \leqq Mt^{n+2}$ uniformly in t and $p \in P$. Integrating by parts we obtain the bound $O(d^{n+\alpha}/\delta^{n+\alpha})$ considered earlier.

The remaining part of (2.14) is

$$\int_K \frac{dz}{|z|^{n+\alpha}} \int_{|y|\leqq 2|p|, y\notin \bar{K}\cup P} |H(y)| \frac{dy}{|p-y|^{n+\beta}}.$$

Since for $z \in K$, $|z|$ and $|p|$ are comparable and, for $z \in K$ and $y \notin \bar{K}$, $|p-y|$ and $|z-y|$ are comparable, this is less than a constant times

$$\int_K dz \int_{|z-y|>c\delta(y)} \frac{|H(y)|}{|y|^{n+\alpha}} \frac{dy}{|z-y|^{n+\beta}}.$$

Adding over m and interchanging the order of integration,

$$\int \frac{H(y)}{|y|^{n+\alpha}} dy \int_{|z-y|>c\delta(y)} \frac{dz}{|z-y|^{n+\beta}} \leqq c \int \frac{|H(y)|}{|y|^{n+\alpha}\delta(y)^{2-\alpha}} dy.$$

This completes the proof of Theorem 1 for $1<\alpha<2$. The argument for $0<\alpha<1$ is somewhat simpler. The analogues of Lemmas 8 and 9 can be found in [4], and those of Lemmas 3 and 5 are clear. The hypothesis (1.1) is not required in Lemma 1 for $0<\alpha<1$ and is therefore not needed in the argument for $\tilde{g}_\varepsilon$. For $\tilde{h}_\varepsilon$ one shows that

$$\int_{E^n} |h(x+z)-h(x)| \frac{dz}{|z|^{n+\alpha}}$$

is finite almost everywhere in P and need not require (1.1).

3. **Proof of Theorem 2.** We will prove Theorem 2 for $1<\alpha<2$ and begin by recalling several lemmas.

LEMMA 1. *Let $u \in L^p$, $1<p<\infty$, and let r be a nonnegative integer. If Ω is a spherical harmonic of degree $m\neq 0$, let*

$$v(x) = \text{p.v.} \int u(x-z) \frac{\Omega(z')}{|z|^n} dz,$$

and let $u(x, \varepsilon)$ and $v(x, \varepsilon)$, $\varepsilon>0$, denote the Poisson integrals of u and v. If

$$(\partial^r/\partial\varepsilon^r)u(x, \varepsilon)$$

has a nontangential limit at every $x \in E \subset E^n$ then so has $(\partial^r/\partial\varepsilon^r)v(x, \varepsilon)$ almost everywhere in E.

Lemma 3 is a special case of Theorem 7, of [9, p. 173].

LEMMA 2. *Let $F \in L^p$, $1 \leqq p < \infty$, and let $F(x, \varepsilon)$ be the Poisson integral of F. Suppose $(\partial^r/\partial\varepsilon^r)F(x, \varepsilon)$ has a nontangential limit at each $x \in E$. Then given $\varepsilon > 0$ there is a closed $P \subset E$, $|E-P| < \varepsilon$, and a splitting $F = G+H$ such that G has an ordinary rth differential almost everywhere and $H = 0$ for $x \in P$.*

For a proof, see [10].

LEMMA 3. *Let $H \in L^p$, $1 \leqq p < \infty$. If for each x in a closed set P, $H(x) = 0$ and*

$$\left(\varepsilon^{-n}\int_{|z|<\varepsilon} |H(x+z) \pm H(x-z)|^p \, dz\right)^{1/p} = O(\varepsilon^r),$$

then for almost all $x \in P$

$$\left(\varepsilon^{-n}\int_{|z|<\varepsilon} |H(x+z)|^p \, dz\right)^{1/p} = o(\varepsilon^r).$$

For a proof see [12, p. 91].

LEMMA 4. *If $f \in T^p_\alpha(x_0)$, $1 < \alpha < 2$, $1 \leqq p < \infty$ then $F = J^\beta f$ $(\alpha+\beta = 2)$ satisfies*

$$\left(\varepsilon^{-n}\int_{|z|<\varepsilon} \left|F(x_0+z) - F(x_0-z) - \sum b_j z_j\right|^p dz\right)^{1/p} = O(\varepsilon^2)$$

for some $b_j = b_j(x_0)$.

Lemma 4 can be proved by the method of [4, pp. 195–197]. Take $x_0 = 0$ and write

$$F(x) - F(-x) = \int f(z)[G_\beta(z-x) - G_\beta(z+x)] \, dz.$$

Thus $F(x) - F(-x)$ differs by a linear term in x from

$$\int \left[f(z) - f(0) - \sum a_j z_j\right][G_\beta(z-x) - G_\beta(z+x)] \, dz.$$

We claim that

$$\int \left[f(z) - f(0) - \sum a_j z_j\right] G_\beta^{(j)}(z) \, dz$$

converges absolutely if

$$\left(\varepsilon^{-n}\int_{|z|<\varepsilon} \left|f(z) - f(0) - \sum a_j z_j\right|^p dz\right)^{1/p} = O(\varepsilon^{n+\alpha}).$$

It is enough to show the part of the integral over $|z| < 1$ converges absolutely. If

$$R(t) = \int_{|z|<t} \left|f(z) - f(0) - \sum a_j z_j\right| dz,$$

$$\int_{|z|<1} \left|f(z) - f(0) - \sum a_j z_j\right| |G_\beta^{(j)}(z)| \, dz \leqq c\int_0^1 \frac{dR(t)}{t^{n-\beta+1}}.$$

That this is finite follows by integrating by parts. Hence $F(x)-F(-x)$ differs by a linear term in x from

$$\int [f(z)-f(0)-\sum a_j z_j]\left[G_\beta(z+x)-G_\beta(z-x)-2\sum x_j G_\beta^{(j)}(z)\right] dz$$
$$= \int_{|z|<2|x|} + \int_{|z|>2|x|} = A(x)+B(x).$$

Here

$$|B(x)| \leqq c\int_{|z|>2|x|} \left|f(z)-f(0)-\sum a_j z_j\right| \frac{|x|^3}{|z|^{n-\beta+3}}\, dz$$
$$= c|x|^3 \int_{2|x|}^{\infty} \frac{dR(t)}{t^{n+\alpha+1}} = O(|x|^2).$$

The terms of $A(x)$ majorized by

$$|x| \int_{|z|<2|x|} \left|f(z)-f(0)-\sum a_j z_j\right| |G_\beta^{(j)}(z)|\, dz = O(|x|) \int_0^{2|x|} \frac{dR(t)}{t^{n-\beta+1}} = O(|x|^2).$$

Hence for $|x|<\varepsilon$,

$$\left|F(x)-F(-x)-\sum b_j x_j\right| \leqq c\varepsilon^2 + \int_{|z|<2\varepsilon} \left|f(z)-f(0)-\sum a_j z_j\right| G_\beta(z+x)\, dz$$
$$+\int_{|z|<2\varepsilon} \left|f(z)-f(0)-\sum a_j z_j\right| G_\beta(z-x)\, dz.$$

For $1\leqq p<\infty$, Young's theorem implies

$$\left(\int_{|x|<\varepsilon} \left|F(x)-F(-x)-\sum b_j x_j\right|^p dx\right)^{1/p}$$
$$\leqq c\varepsilon^{2+n/p} + 2\left(\int_{|z|<2\varepsilon} \left|f(z)-f(0)-\sum a_j z_j\right|^p dz\right)^{1/p} \int_{|z|<3\varepsilon} G_\beta(z)\, dz = O(\varepsilon^{2+n/p}),$$

which proves the lemma.

Let $1\leqq\alpha<2$ and let Ω be a spherical harmonic of degree $m\geqq 0$, $m\neq 1$. In proving Theorems 2 and 3, it will be convenient to use an approximation $f(x,\varepsilon)$ to

$$\tilde{f}_\varepsilon(x) = \int_{|z|>\varepsilon} [f(x-z)-f(x)] \frac{\Omega(z')}{|z|^{n+\alpha}}\, dz$$

which is harmonic in (x,ε) for $x\in E^n$, $\varepsilon>0$. For this purpose we define

$$f(x,\varepsilon) = c_m^{(\alpha)} \int_{E^n} f(x+z)[|z|^\alpha \Omega(z') e^{-\varepsilon|z|}]^\wedge\, dz$$

where $c_m^{(\alpha)}$ is an appropriate constant depending only on α, n and m. The harmonic function $f(x,\varepsilon)$ is considered in [14] where the following facts are proved.

LEMMA 5. *For $f\in L^p$, $1\leqq p<\infty$,*

(a) $f(x,\varepsilon)=\int_{E^n} f(x+z)K(z,\varepsilon)\,dz$,

where

$$K(z,\varepsilon) = \omega_m^{(\alpha)}\nu_m^{(\alpha)}(\varepsilon/|z|)\Omega(-z')|z|^{-n-\alpha},$$

$$\nu_m^{(\alpha)}(r) = \int_0^\infty e^{-rs}s^{\gamma+\alpha+1}J_{m+\gamma}(s)\,ds,$$

$J_\nu(s)$ is the Bessel function of order ν, $\gamma=(n-2)/2$ and $\omega_m^{(\alpha)}$ is a constant depending only on α, n and m;

(b) $|\nu_m^{(\alpha)}(r)|\leqq Ar^{-n-\alpha}$;

(c) $|\omega_m^{(\alpha)}\nu_m^{(\alpha)}(r)-1|\leqq A[(mr)^{1/2}+(mr)^{3/2}]$, $A=A_{\alpha,n}$;

(d) $\int_{E^n} K(z,\varepsilon)\,dz=0$.

The crucial lemma in proving Theorem 2 is

LEMMA 6. *Let $f\in L^p$, $1\leqq p<\infty$, and $f\in T_\alpha^p(x_0)$, $1<\alpha<2$. If $\lim_{\varepsilon\to 0}\tilde{f}_\varepsilon(x_0)$ exists and is finite then $f(x,\varepsilon)$ is bounded in every cone $\{(x,\varepsilon) : |x-x_0|<c\varepsilon\}$.*

Take $x_0=0$ and consider

$$\begin{aligned} f(0,\varepsilon)-\tilde{f}_\varepsilon(0) &= \int_{|z|<\varepsilon} [f(z)-f(0)]K(z,\varepsilon)\,dz \\ &\quad + \int_{|z|>\varepsilon} [f(z)-f(0)]\left[K(z,\varepsilon)-\frac{\Omega(-z')}{|z|^{n+\alpha}}\right]dz \\ &= A_\varepsilon+B_\varepsilon. \end{aligned}$$

Here we have used (d) of Lemma 5. Since $f\in T_\alpha^p(0)$, there are constants a_j, $j=1,\ldots,n$, such that

$$R(t) = \int_{|z|<t} \left|f(z)-f(0)-\sum a_jz_j\right| dz \leqq Mt^{n+\alpha}.$$

Since Ω is orthogonal to polynomials of degree 1 ($m\neq 1$), neither A_ε nor B_ε is changed if we replace $f(z)-f(0)$ in its integrand by $f(z)-f(0)-\sum a_jz_j$. By (b) of Lemma 5,

$$|A_\varepsilon| \leqq c\varepsilon^{-n-\alpha}R(\varepsilon) = O(1),$$

and by (c) of Lemma 5,

$$|B_\varepsilon| \leqq c\int_\varepsilon^\infty \left(\frac{\varepsilon}{t}\right)^{1/2}\frac{dR(t)}{t^{n+\alpha}}.$$

Integrating by parts, B_ε is bounded.

This shows that $f(0,\varepsilon)$ is bounded. To complete the proof, suppose (x,ε) satisfies $|x|<c\varepsilon$ and consider

$$f(x,\varepsilon)-f(0,\varepsilon) = \int [f(z)-f(0)][K(z-x,\varepsilon)-K(z,\varepsilon)]\,dz.$$

Since $\int K(z, \varepsilon)\, dz = \int z_j K(z, \varepsilon)\, dz = 0$([2]), also $\int z_j K(z-x, \varepsilon)\, dz = 0$ and we can majorize the right side above by

$$\begin{aligned}\int_{|z|<2c\varepsilon} \Big|f(z)-f(0)-\sum a_j z_j\Big|(|K(z-x, \varepsilon)|+|K(z, \varepsilon)|)\, dz \\ +\int_{|z|>2c\varepsilon} \Big|f(z)-f(0)-\sum a_j z_j\Big||K(z-x, \varepsilon)-K(z, \varepsilon)|\, dz \\ = A'_\varepsilon + B'_\varepsilon.\end{aligned}$$

As before, A'_ε is bounded. To show B'_ε is bounded, we must estimate the first order derivatives of $K(z, \varepsilon)$ with respect to z. However,

$$\frac{d}{dr}\nu_m^{(\alpha)}(r) = -\int_0^\infty e^{-rs} s^{\gamma+\alpha+2} J_{m+\gamma}(s)\, ds.$$

By an argument like that used for Lemma (1.3) of [14],

$$\frac{d}{dr}\nu_m^{(\alpha)}(r) = O(1)+O(r^s), \qquad s > 0. \tag{3.1}$$

Hence the first order derivatives of $K(z, \varepsilon)$ are bounded by a constant times $\varepsilon^s|z|^{-n-\alpha-1-s}$ for $s \geqq 0$.

Since $|x| < c\varepsilon$,

$$|B'_\varepsilon| \leqq c\varepsilon^s|x| \int_{2c\varepsilon}^\infty \frac{dR(t)}{t^{n+\alpha+1+s}} \leqq c\varepsilon^{s+1}\int_{2c\varepsilon}^\infty \frac{dR(t)}{t^{n+\alpha+1+s}} = O(1).$$

This proves Lemma 6.

In particular, if $f \in L^p$, $1 \leqq p < \infty$, and $f \in T_\alpha^p(x)$ and $\lim_{\varepsilon\to 0} \tilde{f}_\varepsilon(x)$ exists and is finite for $x \in E$, then $f(x, \varepsilon)$ is bounded in each nontangential cone with vertex at a point of E. By a well-known theorem of Calderón (see [1]), $f(x, \varepsilon)$ has a nontangential limit at almost every point of E. If $F = J^\beta f$, we claim this implies that

$$\int_{E^n} F(x+z)[|z|^2\Omega(z')e^{-\varepsilon|z|}]^\wedge\, dz \tag{3.2}$$

has a nontangential limit almost everywhere in E.

For if f is infinitely differentiable and has compact support, (3.2) is

$$\int \hat{F}(z)e^{i(x\cdot z)}|z|^2\Omega(z')e^{-\varepsilon|z|}\, dz = \int \hat{f}(z)\, d\hat{\mu}(z)e^{i(x\cdot z)}|z|^\alpha\Omega(z')e^{-\varepsilon|z|}\, dz$$

by Lemma 2(a) of §2 with $\lambda = \beta$. The last integral is $f_1(x, \varepsilon)$ for the function $f_1 = f * d\mu$, and the same is true for any $f \in L^p$, $1 \leqq p < \infty$, by approximating. Hence (3.2) has a nontangential limit almost everywhere in E if both

(a) $f_1 = f * d\mu \in T_\alpha^p(x)$ and

(b) $\int_{|z|>\varepsilon} [f_1(x-z)-f_1(x)](\Omega(z')/|z|^{n+\alpha})\, dz$ converges for almost all $x \in E$.

([2]) Since $\alpha > 1$, (2) and (3) of Lemma 5 imply $z_j K(z, \varepsilon)$ is integrable.

By Lemma 2 of §2, f_1 differs from f by the sum of a linear combination of terms $J^{2k}f$, $k \geqq 1$, and a term $f * R$ where R has derivatives up to a preassigned order in all L^p, $1 \leqq p \leqq \infty$. Clearly f and $f * R$ satisfy (a) and (b) in E. Now $J^{2k}f \in L_2^p \subset L_\alpha^p$. Hence (b) is true for each $J^{2k}f$ by Lemma 1 of §2. For (a) we use the proof of Lemma (1.5) of [14] for $p > 1$ and Lemma 4 of §2 for $p = 1$.

Now suppose $f \in T_\alpha^p(x)$, $1 \leqq p < \infty$, $1 < \alpha < 2$, for $x \in E$ and each

$$\int_{|z|>\varepsilon} [f(x-z)-f(x)] \frac{\Omega_j(z')}{|z|^{n+\alpha}}\, dz$$

converges for $x \in E$ where $\{\Omega_j\}$ is a normalized basis for the spherical harmonics of a fixed degree $m \neq 1$, $m = 0$ if $p = 1$. With $F = J^\beta f$, each

$$\int_{E^n} F(x+z)[|z|^2\Omega_j(z)e^{-\varepsilon|z|}]^\wedge\, dz$$

has a nontangential limit almost everywhere in E. For smooth F and $m \neq 0$, the last integral is a constant times

$$\frac{\partial^2}{\partial\varepsilon^2} \int_{E^n} (T_jF)(x+z)[e^{-\varepsilon|z|}]^\wedge\, dz$$

where $(T_jF)(x) = \text{p.v. } F * \Omega_j(x)/|x|^n$ (see [3, p. 906]). Since T_j is bounded on L^p for $p > 1$, the same is true for any $f \in L^p$, $p > 1$.

Applying Lemma 1, each

$$\frac{\partial^2}{\partial\varepsilon^2} \int_{E^n} (T_j^2F)(x+z)[e^{-\varepsilon|z|}]^\wedge\, dz$$

has a nontangential limit almost everywhere in E. Since $\sum_j \Omega_j^2$ is constant (see [6, p. 243(2)]), $\sum T_j^2F = F$. Hence

$$\frac{\partial^2}{\partial\varepsilon^2} \int_{E^n} F(x+z)[e^{-\varepsilon|z|}]^\wedge\, dz$$

has a nontangential limit almost everywhere in E. If $m = 0$ $(\Omega \equiv 1)$ the same is true for $1 \leqq p < \infty$.

We now decompose F according to Lemma 2. Theorem 2 will follow if $H \in t_2^p(x)$ for almost all $x \in P$. Since $F = J^\beta f$ and G satisfy the conclusion of Lemma 4 in P, so does H. Since $H = 0$ in P,

$$\left(\varepsilon^{-n} \int_{|z|<\varepsilon} |H(x+z)-H(x-z)|^p\, dz\right)^{1/p} = O(\varepsilon^2)$$

for $x \in P$, and Theorem 2 follows from Lemma 3 of this section.

4. **Proof of Theorem 3.** In this section we will prove Theorem 3 and use the proof to obtain an improvement of Theorem 2. We begin with Theorem 3. Its

proof is similar to that of Theorem 2, but we need a replacement for Lemma 6 of §3.

Hence let $f \in L^p$, $1 \leqq p < \infty$, and suppose

$$\tilde{f}_\varepsilon(x) = \int_{|z|>\varepsilon} [f(x-z)-f(x)] \frac{\Omega(z')}{|z|^{n+1}}\, dz$$

converges at $x = x_0$, where Ω is a spherical harmonic of degree $m \neq 1$, $m = 0$ if $p = 1$. We claim that

$$f(x_0, \varepsilon) = c_m \int_{E^n} f(x_0+z)[|z|\Omega(z')e^{-\varepsilon|z|}]^\wedge\, dz$$

has a limit as $\varepsilon \to 0$. Here we write $c_m = c_m^{(1)}$, $\omega_m = \omega_m^{(1)}$, $\nu_m = \nu_m^{(1)}$.

For taking $x_0 = 0$,

$$\begin{aligned} f(0, \varepsilon) - \tilde{f}_\varepsilon(0) &= \omega_m \int_{|z|<\varepsilon} [f(z)-f(0)]\nu_m\left(\frac{\varepsilon}{|z|}\right) \frac{\Omega(-z')}{|z|^{n+1}}\, dz \\ &\quad + \int_{|z|>\varepsilon} [f(z)-f(0)]\left[\omega_m\nu_m\left(\frac{\varepsilon}{|z|}\right) - 1\right] \frac{\Omega(-z')}{|z|^{n+1}}\, dz \\ &= A_\varepsilon + B_\varepsilon. \end{aligned}$$

We put

$$S(t) = \int_{|z|<t} [f(z)-f(0)]\Omega(-z')\, dz.$$

Then $S(t) = O(t^n)$ as $t \to \infty$ and, since $\tilde{f}_\varepsilon(0)$ converges, $S(t) = o(t^{n+1})$ as $t \to 0$. However,

$$A_\varepsilon = \int_0^\varepsilon S'(t)G_\varepsilon(t)\, dt$$

and

$$B_\varepsilon = \int_\varepsilon^\delta S'(t)H_\varepsilon(t)\, dt + o(1)$$

for fixed $\delta > \varepsilon$ by Lemma 5(c) of §3. Here of course

$$G_\varepsilon(t) = \omega_m\nu_m(\varepsilon/t)t^{-n-1}, \qquad H_\varepsilon(t) = [\omega_m\nu_m(\varepsilon/t) - 1]t^{-n-1}.$$

Integrating by parts and applying (b) and (c) of Lemma 5 of §3,

$$A_\varepsilon = -\int_0^\varepsilon S(t)G'_\varepsilon(t)\, dt + o(1), \qquad B_\varepsilon = -\int_\varepsilon^\delta S(t)H'_\varepsilon(t)\, dt + o(1).$$

To show A_ε and B_ε tend to zero, it is therefore enough to show that for $s > 0$

$$G'_\varepsilon(t) = O(\varepsilon^{-n-1}t^{-1}), \qquad H'_\varepsilon(t) = O(\varepsilon^s t^{-n-2-s}).$$

The estimate for H'_ε follows from (3.1) and Lemma 5(c) of §3. $G'_\varepsilon(t)$ is a combination of $\nu_m(\varepsilon/t)t^{-n-2}$ and $(d/dt)[\nu_m(\varepsilon/t)]t^{-n-1}$. The first of these is $O(\varepsilon^{-n-1}t^{-1})$ and the second is a constant times

$$\varepsilon t^{-n-3} \int_0^\infty e^{-(\varepsilon/t)s} s^{\gamma+3} J_{m+\gamma}(s)\, ds \leqq c\varepsilon t^{-n-3} \int_0^\infty e^{-(\varepsilon/t)s} s^{2\gamma+3}\, ds,$$

since $|J_{m+\gamma}(s)| \leqq cs^{\gamma}$ (see [14], Lemma (1.2)). Changing variables our claim follows —i.e., $f(0, \varepsilon)$ has a limit as $\varepsilon \to 0$.

Suppose in addition that

$$\left(\varepsilon^{-n} \int_{|z|<\varepsilon} |f(z)+f(-z)-2f(0)|^p \, dz\right)^{1/p} = O(\varepsilon). \tag{4.1}$$

Consider

$$f(x, \varepsilon) = \int f(z)K(z-x, \varepsilon)\, dz$$

where K is defined by Lemma 5 of §3 for $\alpha=1$. If m is odd then $K(z, \varepsilon)$ is odd in z and

$$f(-x, \varepsilon)-f(x, \varepsilon) = \frac{1}{2}\int_{E^n} [f(z)+f(-z)-2f(0)][K(z+x, \varepsilon)-K(z-x, \varepsilon)]\, dz. \tag{4.2}$$

If m is even then $K(z, \varepsilon)$ is even in z and

$$\begin{aligned} &f(x, \varepsilon)+f(-x, \varepsilon)-2f(0, \varepsilon) \\ &\quad = \frac{1}{2}\int_{E^n} [f(z)+f(-z)-2f(0)][K(z+x, \varepsilon)+K(z-x, \varepsilon)-2K(z, \varepsilon)]\, dz. \end{aligned} \tag{4.3}$$

Using (4.1) and arguing as in the last part of the proof of Lemma 6 above, both (4.2) and (4.3) are bounded in any cone $\{(x, \varepsilon) : |x| < c\varepsilon\}$. If, in particular, f satisfies (4.1) for $x \in E$ and $\tilde{f}_\varepsilon(x)$ converges for $x \in E$ then, taking subsets of E, we may assume that

(a) $f(x, \varepsilon)$ is uniformly bounded in (x, ε) for $x \in E$, $0<\varepsilon<\eta$,

(b) either $f(x+z, \varepsilon)+f(x-z, \varepsilon)$ or $f(x+z, \varepsilon)+f(x-z, \varepsilon)-2f(x, \varepsilon)$ is uniformly bounded for $x \in E$ and $|z|<\varepsilon$.

By a simple argument, it follows $f(x, \varepsilon)$ is bounded in some cone with vertex at each point of density of E, and so $f(x, \varepsilon)$ has a nontangential limit almost everywhere in E. Under the hypothesis of Theorem 3, therefore, each

$$\int_{E^n} f(x+z)[|z|\Omega_j(z')e^{-\varepsilon|z|}]^\wedge \, dz$$

has a nontangential limit almost everywhere in E. If $1<p<\infty$ and $m\neq 0$, it follows from Lemma 1 above as before that

$$\frac{\partial}{\partial\varepsilon}\int_{E^n} f(x+z)[e^{-\varepsilon|z|}]^\wedge \, dz$$

has a nontangential limit almost everywhere in E. If $m=0$, the same is true for $1 \leqq p < \infty$.

By Lemma 2 of §3, there is for $\varepsilon>0$ a closed $P\subset E$, $|E-P|<\varepsilon$, and a splitting $f=g+h$ such that $g \in t_1^p(x)$ for almost all x and $h=0$ in P. Since f and g satisfy (4.1) so does h and Theorem 3 follows from Lemma 3 above.

Finally, we remark that the proof just given can be modified to prove Theorem 2 under an apparently weaker hypothesis on f. In fact, the conclusion of Theorem 2 is valid if we replace the hypothesis that $f \in T_\alpha^p(x)$, $x \in E$ by the condition

$$\text{(i)} \qquad \left(\varepsilon^{-n}\int_{|z|<\varepsilon} |f(x+z)+f(x-z)-2f(x)|^p\,dz\right)^{1/p} = O(\varepsilon^\alpha)$$

if $0<\alpha<1$ or

$$\text{(ii)} \qquad \left(\varepsilon^{-n}\int_{|z|<\varepsilon} \left|f(x+z)-f(x-z)-2\sum a_j(x)z_j\right|^p dz\right)^{1/p} = O(\varepsilon^\alpha)$$

if $1<\alpha<2$, $x \in E$.

We note here that Lemma 4 of §3 remains true if the hypothesis $f \in T_\alpha^p(x_0)$ is replaced by (ii) above for $x=x_0$. If instead (i) holds for $x=x_0$ its analogue is

$$\left(\varepsilon^{-n}\int_{|z|<\varepsilon} |F(x+z)+F(x-z)-2F(x)|^p\,dx\right)^{1/p} = O(\varepsilon), \qquad F = J^{1-\alpha}f.$$

An unpublished result of Stein states that if (ii) holds for $\alpha=1$ and each $x \in E$ then $f \in t_1^p(x)$ for almost all $x \in E$. Hence assuming (ii) for $\alpha=1$ does not lead to a strengthening of Theorem 2.

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